4. CLASSES OF RINGS

§4.1. Classes of Rings

Normally we associate, with any property, a set of objects that satisfy that property. But problems can arise when we allow sets to be elements of larger sets in an uncontrolled way. For example the Russell Paradox comes from considering the 'set' $S = \{x \mid x \notin x\}$ and asking whether $S \in S$. If $S \in S$ then S satisfies the property that defines S, that is, $S \notin S$, a contradiction. But if $S \notin S$ it does satisfy the defining property for S and so $S \in S$, again a contradiction.

Constructions such as the set of all sets are just not on, and if we were pursuing axiomatic set theory we'd have to lay down certain axioms to restrict which properties are allowed to give rise to a set. (See my notes on Set Theory for a way to do this.)

But, we're doing ring theory and we want to turn adjectives that describe properties of rings into nouns. We use the word 'class' in place of 'set' to avoid these logical problems. Now it's not the fact that we're using a different word that avoids these logical difficulties. It's the fact that, while sets are allowed to be elements of other sets, classes are not. So we'll talk about the class of all rings, the class of all nilpotent rings etc. The only assumption that we make is that if a certain ring belongs to a class, all isomorphic copies of it also belong to it as well. In other words, in this context, we consider all rings isomorphic to a single one as being identical. We’re not concerned with the nature of the underlying set, but rather the structure of the ring.

As well as having a rich supply of classes to play with we want to consider certain recipes for obtaining a new class from a given one. We define these as ‘class operators’. Essentially they’re functions on the class of all classes of all rings, but as we’ve said, classes are not allowed to be considered as elements of other classes. So a class operator is not to be considered as a set of ordered pairs, or even a class of ordered pairs – it’s simply a rule that associates with each class $\mathscr{X}$ a certain class $\Phi\mathscr{X}$. We shall write operators on the left but avoid the usual $\Phi(\mathscr{X})$ notation as a way of reminding ourselves that these operators are not exactly functions in the usual sense in set theory.

Here are some examples of classes that we’ll be considering:

$\mathscr{B}$	finite rings;
$\mathscr{C}$	commutative rings;
$\mathscr{D}$	integral domains.
$\mathscr{E}$	Euclidean rings
$\mathscr{F}$	fields
$\mathscr{N}$	nil rings
$\mathscr{P}$	nilpotent rings
$\mathscr{S}$	simple rings
$\mathscr{T}$	torsion rings (every element having finite additive order);

A **class operator A** is a rule that creates a ring class **A**$\mathfrak{X}$ from a class $\mathfrak{X}$. If a ring R belongs to a class $\mathfrak{X}$ we say that R is an $\mathfrak{X}$-ring. If A$\mathfrak{X} \subseteq \mathfrak{X}$ (subclasses being defined in exactly the same way as subsets) we say that $\mathfrak{X}$ is **A-closed**.

Here are some class operators that will be useful.

S$\mathfrak{X}$	subrings of $\mathfrak{X}$-rings
Q$\mathfrak{X}$	quotients of $\mathfrak{X}$-rings
P$\mathfrak{X}$	**poly** $\mathfrak{X}$ rings, that is, rings R with an ideal I such that both I and R/I are $\mathfrak{X}$-rings.
$\cup\mathfrak{X}$	unions of ascending chains of $\mathfrak{X}$-ideals of some ring;
$\Sigma\mathfrak{X}$	rings of the form $I + J$ where I, J are $\mathfrak{X}$-subrings of some ring R and I is an ideal of R. (This includes directs sums of $\mathfrak{X}$-rings.)

In Σ-closure, the reason for insisting on one of the summands of I + J being an ideal of a larger ring is to ensure that I + J is indeed a ring. In general the sum of two subrings is not closed under multiplication. But if I is an ideal and $x_1, x_2 \in$ I and $y_1, y_2 \in$ J then

$$(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2.$$

If I, J were merely subrings we'd have $x_1x_2 \in$ I and $y_1, y_2 \in$ J but we wouldn't know about the other two terms. But if I is an ideal of a ring R that contains both I and J then both x_1y_2 and y_1x_2 are in I.

Example 1: $\mathfrak{N}$ is S-closed since every subring of a nil ring is a nil ring.

A construction that produces new classes operators out of existing ones is multiplication of classes operators. We define the **product** of class operators A, B by defining (**AB**)$\mathfrak{X}$ = A(B$\mathfrak{X}$).

Example 2:
(SQ)$\mathfrak{X}$ is the class of all rings that are subrings of quotients of $\mathfrak{X}$-rings.
(QS)$\mathfrak{X}$ is the class of all rings that are quotients of subrings of $\mathfrak{X}$-rings.
$S^2 = S$, since a subring of a subring of a ring, R, is a subring of R and every subring of R is a subring of a subring of R since R is a subring of itself.
Similarly $Q^2 = Q$.

Multiplication of class operators is clearly associative and, as is usual, when we define any sort of associative product, we can define powers. So we define $A^0\mathfrak{X} = \mathfrak{X}$; $A^2\mathfrak{X} = A(A\mathfrak{X})$ and so on. Moreover we define $A^\infty\mathfrak{X}$ to be the union of the $A^n\mathfrak{X}$.

We now investigate which of our collection of classes are closed under each of our collection of class operators.

S-CLOSURE:

𝔅 is clearly S-closed: every subring of a finite ring is finite.

𝒞 is clearly S-closed: every subring of a commutative ring is commutative.

𝔇 is NOT S-closed: For example Z is an integral domain but 2Z is not (no identity).

𝔈 is NOT S-closed: same counter-example.

𝔉 is NOT S-closed: Z is a subring of the field R but is not a field.

𝔑 is S-closed: every subring of a nil ring is a nil-ring.

𝒫 is S-closed: every subring of a nilpotent ring is nilpotent.

𝔖 is NOT S-closed: The ring of 2×2 matrices over a field is simple. Yet the subring consisting of all diagonal matrices is not.

𝔗 is S-closed: every subring of a torsion ring is a torsion ring.

In general, the class of rings that has all of its elements satisfying some property, is clearly S-closed.

Q-CLOSURE:

𝔅 is Q-closed: Every quotient of a finite ring is finite.

$\mathcal{C}$ **is Q-closed:** Every quotient of a commutative ring is commutative.

$\mathcal{D}$ **is NOT Q-closed:** Z is an integral domain but Z/4Z is not.

$\mathcal{E}$ **is NOT Q-closed:** Same counter-example.

$\mathcal{F}$ **is NOT Q-closed:** Fields only have two ideals, themselves and {0}. The quotient F/F only has one element and the smallest field has order 2.

$\mathcal{N}$ **is Q-closed:** Every quotient of a nil ring is a nil ring.

$\mathcal{P}$ **is Q-closed:** Every quotient of a nilpotent ring is a nilpotent ring.

$\mathcal{S}$ **is not Q-closed:** Simple rings only have two ideals, themselves and {0}. The quotient R/R only has one element and the smallest simple ring has order 2.

$\mathcal{T}$ **is Q-closed:** If R is torsion then R/I is torsion. That is, if $nr = 0$ then $n(r + \mathrm{I}) = \mathrm{I}$ for any ideal I.

In general, if the trivial ring, {0}, is not an $\mathcal{X}$-ring the class $\mathcal{X}$ is not Q-closed, because R/R is the trivial ring for all rings R.

P-CLOSURE

$\mathcal{B}$ **is P-closed:** If R has an ideal of order m, and R/I has order n then R has order mn.

$\mathcal{C}$ **is NOT P-closed:** For example if R = $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, meaning the set of all matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and I = $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, then

I is an ideal of *R*. But, although both I and R/I are commutative rings, R is not.

$\mathfrak{D}$ is NOT P-closed: R = Z ⊕ Z has an ideal, I, that’s isomorphic to Z where R/I is also isomorphic to Z. But R does not satisfy the Cancellation Law.

$\mathfrak{E}$ is NOT P-closed: Z ⊕ Z is a counter-example.

$\mathfrak{F}$ is NOT P-closed: Fields only have two ideals, themselves and {0}. The quotient F/F only has one element and the smallest field has order 2.

$\mathfrak{N}$ is P-closed: Suppose R/I and I are nil. If $x \in$ R, then for some n, $(x + I)^n = x^n + I = I$, and so $x^n \in$ I. Hence $x^{nm} = (x^n)^m = 0$ for some m.

$\mathfrak{P}$ is P-closed: Suppose R/I and I are nilpotent. Then $R^n \leq$ I for some n. Hence, for some m, Since I is nilpoyent, $I^m = 0$ for some m. Hence $(R^n)^m \leq I^m = 0$.

$\mathfrak{S}$ is not P-closed: R = Q ⊕ Q has an ideal, I, that’s isomorphic to Q where R/I is also isomorphic to Q. So I and R/I are simple, but R is not,

$\mathfrak{T}$ is Q-closed: Suppose R/I and I are torsion. If x $\in$ R, then for some *n*, n(x + I) = nx + I = I, and so nx $\in$ I. Hence (mn)x = m(nx) = 0 for some m.

$\cup$-CLOSURE:

Theorem 1: A property that can be expressed in terms of finitely many elements is will hold in the union of an ascending chain of ideals if it holds in each member of the chain.

Proof: If x_1, x_2, …, x_k are in the union of the ascending chain of ideals

$$R_0 \le R_1 \le R_2 \le \ldots\ldots.. \le R$$

then they all belong to some R_N. ✋☺

$\mathcal{B}$ **is not ∪-closed:** The zero ring on the Prüfer group (Example 5, Chapter 1) is infinite. Yet it's the union of the ascending chain

$$\langle a_0\rangle \le \langle a_1\rangle \le \langle a_2\rangle \le \ldots$$

all of which are finite.

$\mathcal{C}$ **is ∪-closed &** $\mathcal{T}$ **is ∪-closed:** Follows from Theorem 1.

$\mathcal{D}$ **is ∪-closed:** The identities in an ascending chain of ideals, all of which are integral domains, are the same, by Theorem 6 of chapter 1, and hence it will be the identity for the union. The Commutative Law and the Cancellation Law can each be expressed in terms of two elements, and so by Theorem 1 will hold in the union.

$\mathcal{E}$ **is ∪-closed:**

$\mathcal{F}$ **is ∪-closed:** All of the field axioms can be expressed in terms of finitely many elements and so will apply to the union of an ascending chain of ideals if they hold in each member of the chain.

$\mathcal{N}$ **is ∪-closed:** Follows from Theorem 1.

$\mathcal{P}$ **is NOT ∪-closed:**

$\mathcal{S}$ **is ∪-closed:**

$\mathcal{T}$ **is ∪-closed:** Follows from Theorem 1.

Σ-CLOSURE:

$\mathfrak{B}$ is Σ-closed: If I and J are finite subrings of R, and I is an ideal, then |R + S| ≤ |R|.|S| and so is finite.

$\mathcal{C}$ is NOT Σ-closed: Certainly the direct sum of two commutative rings is commutative, but Σ-closure is stronger than just closure under direct sums.

For example $I = \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$ are commutative ideals of $R = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$.

Here * is a wildcard so, for example, I = the set of all matrices of the form $\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for all a, b. Yet R = I + J is not commutative.

$\mathfrak{D}$ is NOT Σ-closed: Let R = Z ⊕ Z.

$\mathfrak{E}$ is NOT Σ-closed: Same counter-example.

$\mathfrak{F}$ is NOT Σ-closed:R = Q ⊕ Q is a counter-example.

$\mathfrak{N}$ is Σ-closed: We shall show that any class that's P- and Q-closed is also Σ-closed.

$\mathcal{P}$ is Σ-closed:

$\mathfrak{S}$ is NOT Σ-closed: R = Q ⊕ Q is a counter-example.

$\mathfrak{T}$ is Σ-closed: Suppose that I and J are torsion subrings of R, and I is an ideal of R.

Then I + J is a subring of R.

If x = r + s ∈ R then *m*r = 0 and *n*s = 0 for some positive integers *m*, *n*.
Hence *mn*x = *n*(*m*r) + *m*(*n*s) = 0.

We can summarise our results in a table.

	$\mathscr{B}$	$\mathscr{C}$	$\mathscr{D}$	$\mathscr{E}$	$\mathscr{F}$	$\mathscr{N}$	$\mathscr{P}$	$\mathscr{S}$	$\mathscr{T}$	$\mathscr{Z}$
S	√	√	×	×	×	√	√	×	√	√
Q	√	√	×	×	×	√	√	×	√	√
P	√	×	×	×	×	√	√	×	√	×
∪	×	√	√	?	√	√		?	√	√
Σ	√	×	×	×	×	√	√	×	√	×
⊕	√	√	×	×	×	√	√	×	√	√

Theorem 2: Closure under both P and Q implies ∑-closure.
Proof: Suppose I, J ∈ $\mathscr{X}$ with J a 2-sided ideal of R.
Then (I + J)/J ≅ I/(I ∩ J) ∈ $\mathscr{X}$ and hence I + J ∈ $\mathscr{X}$. ✋☺

§4.2. Zorn’s Lemma

The rings we’re interested in are infinite and infinite sets can have some strange properties. Given a certain property we want to find the largest ideal that has that property. But such a largest might not exist. One reason why a set fails to have a largest is because there may be more than one possible candidate. For example we might ask which proper subspace of $\mathbb{R}^3$ is largest. There are plenty of 2-dimensional subspaces but none contains any of the others. The other, and more subtle reason why a set may not have a largest is that there may be larger and larger elements, but no largest. For example there's no largest real number x for which $x^2 < 2$. We shall focus on the second problem.

This will take us into the rarefied world of Axiomatic Set Theory where we discuss Zorn's Lemma. It isn't our role here to give a complete account of the necessary set theory, but, before we start using Zorn's Lemma, we need some discussion as to its logical status.

A **partially-ordered set** is a set X, together with a relation (we shall denote it by ≤, though keep in mind that it may not bear any relation to any ordering we've ever met). But, like ordinary "less-than-or-equals" we do insist that it have the following three properties:

(1) $x \leq x$ for all $x \in X$ (the reflexive property);
(2) if $x \leq y$ and $y \leq x$ then $x = y$ (the anti-symmetric property);
(3) if $x \leq y$ and $y \leq z$ then $x \leq z$ (the transitive property).

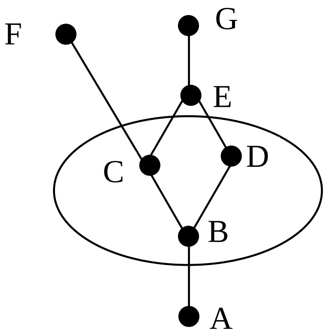
An example of such an ordering is that of "subset of". Now notice that "subset of" fails to have a fourth property that we expect of ordinary ≤:

(4) for all $x, y \in X$ either $x \leq y$ or $y \leq x$.

We can easily have subsets where neither is a subset of the other. This fourth property is not required for a set to be partially ordered. But if it *does* hold we say that the set is **totally ordered**, or simply that the set is a **chain**.

By analogy with ordinary ≤ of numbers we define **≥**, **<** and **>**. If X is a partially ordered set, and $S \subseteq X$ then $b \in X$ is an **upper bound** for S if $x \leq b$ for all $x \in S$. There's no need for the upper bound to be in the set, but if it *is* it is called a **greatest** element. A set may not have a greatest element, but it may have maximal elements. These are elements where nothing in the subset is bigger. A **maximal element** of S is thus an element $m \in S$ such that if $x \geq m$ for some $x \in S$ then $x = m$. Finally we can combine a couple of these together and talk of a **least upper bound**. Similarly we define **lower bound**, **least element**, **minimal element** and **greatest lower bound**.

Example 4: Consider the set {A, B, C, D, E, F, G} where the ≤ ordering is described by the following diagram ($x \leq y$ if and only if you can get from x to y in the diagram by a path consisting of one or more edges, each traversed in an upwards direction).



Let S = {B, C, D}. Then E and G are upper bounds for S, with E the least upper bound. But S has no largest element, though C, D are maximal elements. S has a least element, B and this is the only minimal element. A, B are both lower bounds, but B is the greatest lower bound.

In finite partially ordered sets there will always be at least one maximal element. If x is not maximal there's an element y with $y \geq x$. If y isn't maximal then there's an element z with $y \leq z$, and so on. With finitely many elements we must eventually reach a maximal element.

On the other hand, with infinite partially ordered sets, there may be no maximal element. For example consider the set of real numbers is partially ordered in the usual way. The subset $\{x \mid 0 \leq x^2 < 2\}$ has no maximal element. But it does have a least upper bound, $\sqrt{2}$. If we restricted ourselves to just rational numbers there will not even be a least upper bound.

ZORN'S LEMMA: If every chain in the non-empty partially ordered set X has an upper bound then X has a maximal element.

'Proof': A pseudo proof of this goes along the following lines. Take an element $x_1 \in X$. If x_1 isn't maximal there's an element x_2 that's bigger than x_1. If that isn't maximal there's an x_3 bigger than x_2, and so on. So if X doesn't have a maximal element there must be a chain $x_1 < x_2 < \ldots$ in X. By our assumption, every chain in X has an upper bound, and so there exists y_1 that's bigger than all of the x_i. By the same argument we can produce a chain of y_i's so that $x_1 < x_2 < \ldots < y_1 < y_2 < \ldots$ So far so good. The pseudo nature of the pseudo proof comes when we say "surely the process must eventually terminate".

We haven't called Zorn's Lemma a theorem because it isn't one. There is a proof that it is impossible to either prove or disprove Zorn's Lemma. This is an example of an undecidable proposition in set theory. We set up a collection of axioms that attempt to avoid a contradiction like the Russell Paradox. Now if you were to set up a random collection of axioms it's possible that they would be self-contradictory. The simplest way to show that they're consistent is to give a concrete example that satisfies them all. The group axioms are consistent because we can give examples of groups.

But when it comes to the axioms of set theory, how are we going to produce an example of a universe of sets satisfying the axioms without making use of sets in order to construct such a model? So it would seem that we'll probably never be able to prove that any set of axioms for set theory is consistent.

What has been proved is that *if* the axioms of set theory are consistent, and therefore that there is a model that satisfies them, then it is possible to construct within it another model that satisfies all the set theory axioms together with Zorn's Lemma. This proves that Zorn's Lemma is *consistent* with the other axioms of set theory. If ever a contradiction is reached after using Zorn's

Lemma, the problem would lie with the basic set theory axioms and not with Zorn's Lemma.

On the other hand, again assuming that a model for set theory exists, we can construct within it a different model that satisfies the basic axioms of set theory but in which Zorn's Lemma is false! This proves that Zorn's Lemma is *independent* from the other axioms of set theory. It can never be proved from them, otherwise such a model couldn't exist.

So Zorn's Lemma isn't a theorem – it isn't even a lemma – it is in fact an optional extra axiom that we are logically free to accept or reject. There are other axioms that are equivalent to Zorn's Lemma. The most famous of these is the Axiom of Choice. At least this one goes by the correct name of "axiom".

Axiom of Choice: If X is a set of non-empty sets, and Y is the set of all the elements of the elements of X, there is a function $\Phi{:}X \to Y$ such that $\Phi(x) \in x$.

Here X is a collection of non-empty sets. The task is to choose one from each. The choices don't have to be different. Some of the sets might overlap and we might choose the same element from many of these sets. The choice is via a choice function, which assigns to each of the sets the chosen element from that set.

Example 5: Let $X_1 = \{1, 2, 3\}$, $X_2 = \{1, 5, 9, 10\}$, X_3 = the set of prime numbers.

Let $X = \{X_1, X_2, X_3\}$. Then Y consists of all the prime numbers, together with 1, 9 and 10. An example of a function described in the Axiom of Choice is:

$$\Phi(X_1) = 2,\ \Phi(X_2) = 10,\ \Phi(X_3) = 17.$$

It may well seem to you that the Axiom of Choice is obvious. If we have a bunch of non-empty sets surely we can choose one element from each! Certainly if the sets are finite and there are finitely many of them it is not only obvious, it is in fact true. But making infinitely many arbitrary choices is a bit different.

It has been shown that if we assume the Axiom of Choice we can prove Zorn's Lemma, and vice versa. The Axiom of Choice therefore has the status of an optional axiom, equivalent to Zorn's Lemma. We are logically free to assume the Axiom of Choice, but if we do we must also accept Zorn's Lemma. We are logically free to deny the Axiom of Choice, but if we do we must deny Zorn's Lemma.

What are the practical consequences of deciding to accept the Zorn's Lemma/Axiom of Choice package? In terms of the truth of specific examples there can be none. No bridge will fall down because the engineer based his mathematics on the Axiom of Choice. The differences are more aesthetic. Assuming the Axiom of Choice and Zorn's Lemma will allow, in some cases, theorems that are cleaner and simpler to state and in some cases easier to prove, though the specific consequences of mathematics with or without it will be identical. In the following we shall assume Zorn’s Lemma.

§4.3. Radical Classes

A class of rings is a **radical class** if it is closed under Q, $\cup$ and P, and hence also under Σ. Being Q-closed every radical class must contain the trivial ring, 0.

Theorem 3: If $\mathcal{X}$ is a radical class then every ring has a largest $\mathcal{X}$-ideal, $\mathbf{R}_{\mathcal{X}}$, called the
$\mathcal{X}$-radical of R.

Proof: The set of $\mathcal{X}$-ideals is a partially ordered set under $\subseteq$. Since $\mathcal{X}$ is $\cup$-closed, every chain, C, has an upper bound, the union of all the sets in the chain, and so by Zorn’s Lemma there’s a maximal element M. If I is any $\mathcal{X}$-ideal then M + I is an $\mathcal{X}$-ring so $I \leq M$. Hence M is the union of all the $\mathcal{X}$-ideals. 🖐☺

Theorem 4: If $\mathcal{X}$ is a radical property $(R \oplus S)_{\mathcal{X}} = R_{\mathcal{X}} \oplus S_{\mathcal{X}}$.

Proof: The ideals of $R \oplus S$ are $I \oplus J$ where I, J are ideals of R, S respectively. 🖐☺

If $\mathcal{X}$ is a radical class, a ring R is defined to be $\mathcal{X}$-**semisimple** if its $\mathcal{X}$-radical is zero. The class of $\mathcal{X}$-semisimple rings is denoted by $\mathcal{X}^{\perp}$. Clearly $\mathcal{X} \cap \mathcal{X}^{\perp} = \{0\}$, that is the only ring that is an $\mathcal{X}$-ring and $\mathcal{X}$-semisimple is the trivial ring. Most rings are neither $\mathcal{X}$-rings nor $\mathcal{X}$-semisimple, but of course simple rings have to be one or the other – the $\mathcal{X}$-radical of a simple ring R is either R or 0.

If we factor out the $\mathcal{X}$-radical of any ring we will be left with an $\mathcal{X}$-semisimple ring. This might seem obvious, but remember it doesn't work for rings that are not poly closed.

Theorem 5: $R/R_{\mathcal{X}}$ is $\mathcal{X}$-semisimple.

Proof: Suppose $(R/R_{\mathcal{X}})_{\mathcal{X}} = S/R_{\mathcal{X}}$. Since $\mathcal{X}$ is P-closed, $S \in \mathcal{X}$ so $S = R_{\mathcal{X}}$. ✋☺

Theorem 6: A class of rings $\mathcal{X}$ is a radical class if and only if it is Q-closed and every ring R contains an $\mathcal{X}$-ideal $R_{\mathcal{X}}$ containing all $\mathcal{X}$-ideals with $R/R_{\mathcal{X}} \in \mathcal{X}^{\perp}$.

Proof: Suppose $\mathcal{X}$ is a class satisfying these properties.
We must show that it is both $\cup$- and P-closed. Suppose R/I and I are in $\mathcal{X}$. Then $I \le R_{\mathcal{X}}$.
Now, $R/R_{\mathcal{X}} \cong (R/I)/(R_{\mathcal{X}}/I) \in \mathcal{X}$ so $R_{\mathcal{X}} = R$, that is, $R \in \mathcal{X}$.

Suppose that $R_1 \le R_2 \le \ldots$ is an ascending chain of $\mathcal{X}$-ideals of a ring R and let $U = \cup R_i$.

For each n, $R_n \le U$ and $(U_{\mathcal{X}} + R_n)/U_{\mathcal{X}} \cong R_n/(R_n \cap U_{\mathcal{X}} \in \mathcal{X}$. so $R_n \le U_{\mathcal{X}}$..

Hence $U = U_{\mathcal{X}}$ so $U \in \mathcal{X}$. ✋☺

§4.4. The Nil Radical

Theorem 7: The class, $\mathcal{N}$, of nil rings is a radical class.

Proof: $\mathcal{N}$ is clearly Q-closed. If $I_1 \le I_2 \le \ldots$ is an ascending chain of nil ideals then $\cup I_r$ is clearly nil, for if $x \in \cup I_r$, $x \in I_r$ for some r. Since I_r is nil, $x^n = 0$ for some n.

Finally we must show that N is poly-closed.

Suppose that I is an ideal of R and both I and R/I are nil rings.

Let $x \in R$. Since R/I is nil, $(r + I)^n = r^n + I = I$, for some n.

Thus $r^n \in I$. Since I is nil, $(r^n)^m = 0$ for some m. ✋☺

The class of nilpotent rings is probably a more useful class than the class of nil rings, but unfortunately it isn't a radical class.

Theorem 8: The class of nilpotent rings is closed under P, Q and Σ, but not under ∪.

Proof: Consider Example 9 in Chapter 1, the semigroup ring $\mathbb{R}X$ where

$$X = \{a_x \mid x \in \mathbb{Q} \text{ and } 0 < x < 1\}.$$

$$\text{and } a_x\, a_y = \begin{cases} a_{x+y} \text{ if } x + y < 1 \\ 0 \text{ if } x + y \geq 1 \end{cases}.$$

But for each positive integer n, let $R_n = \left\{ \sum_{1/n<x<1} \lambda_x a_x \mid \lambda_x \in \mathbb{R} \right\}$.

These are nilpotent ideals since $(R_n)^n = 0$. Moreover $R_2 < R_3 < R_4 < \ldots$ is an ascending chain of these nilpotent ideals, with $\cup\{R_k\} = R$. Yet R not nilpotent since, for all n:

$$a_{1/2}\, a_{1/4}\, a_{1/8} \ldots a_{1/2^n} = a_{1-2^n} \neq 0.$$

Example 6: Let R be the set of all $n \times n$ upper triangular matrices over a field F.

The nil-radical $R_{\mathfrak{N}}$ consists of those matrices with 0's on the diagonal and $R/R_{\mathfrak{N}} \cong F \oplus \ldots \oplus F$ (n copies).

Theorem 9: If R is a commutative ring the set of all nilpotent elements is an ideal and is thus the nil radical of R.

Proof: Let a, b be elements of the commutative ring R. If $a^m = b^n = 0$ then $(a + b)^{m+n-1} = 0$ and $(ab)^{\min(m,n)} = 0$. Commutativity is needed since in the ring of 2×2 matrices $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are nilpotent yet their sum and product aren't.

Theorem 10 (MACHKE'S THEOREM): Let G be a finite group of order n and suppose that the characteristic of F doesn't divide n. Then the group ring FG is nil semi-simple.

Proof: Suppose $0 \neq S$ is a nil ideal and let $0 \neq x = \Sigma a_g g \in S$.

Since S is an ideal we may suppose, without loss of generality, that $a_1 = 1$.

Let ρ:FG$\rightarrow$ End$_F$FG be the ring homomorphism defined by $x(y\rho) = xy$.

Since x is nilpotent, so is $x\rho$ and hence trace$(x\rho) = 0$.

Now trace$(g\rho) = 0$ for $g \neq 1$ and trace$(1) = n$.

Hence $n1 = 0$ so the characteristic of F divides n.

§4.5. The Jacobson Radical

Define a ring to be a **Jacobson Ring** if every element is right quasi-regular. Let $\mathcal{J}$ be the class of rings in which every element is right quasi-regular.

Theorem 13: $\mathcal{J}$ is a radical class.

Proof: P-closure follows from the associativity of $\circ$.

Example 7: Let $\Omega = \left\{\dfrac{\text{even}}{\text{odd}}\right\}$ be the subring of $\mathbb{Q}$ defined in Example 6 of Chapter 1.

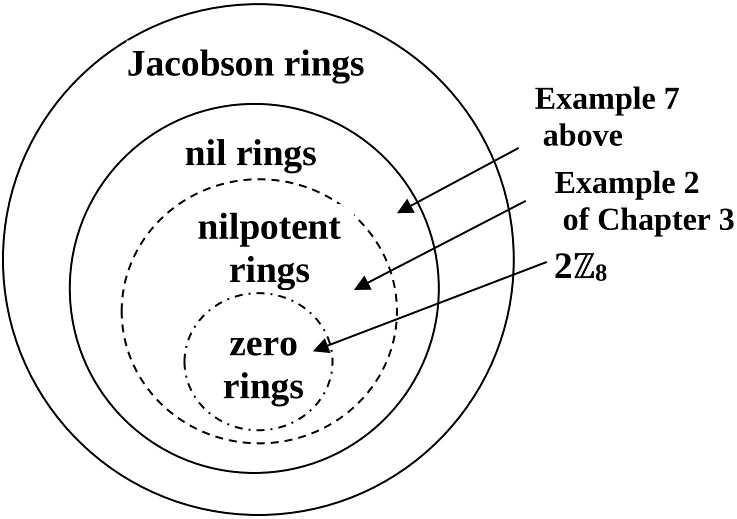
Clearly $\Omega \notin \mathcal{N}$. Let $x \in \Omega$. Then, for some integers m, n,

$$x = \frac{2m}{2n + 1}\,.$$

Let $y = \dfrac{-2m}{2m + 2n + 1} \in \Omega$. Then $x \circ y = 0$.

Hence $\Omega \in \mathscr{J} - \mathscr{N}$ and so $\mathscr{N}$ is a proper subclass of $\mathscr{J}$.

The following diagram shows the relationship between several classes of rings.



The solid lines represent the radical classes. These classes are all distinct.

§4.6. Upper and Lower Radical Properties

We have used two closely related radical classes, nil rings and Jacobson rings. In this section we consider radical classes in general and three class operators in particular. Let $\mathbb{S}$ be the class of all simple rings.

H$\boldsymbol{\mathfrak{X}}$ = {R | every non-zero quotient of *R* has a non-zero ideal in $\mathscr{X}$}.
I$\boldsymbol{\mathfrak{X}}$ = {R | every non-zero ideal of *R* has a non-zero quotient in $\mathscr{X}$}.
N$\boldsymbol{\mathfrak{X}}$ = {R | *R* has no non-zero quotient in $\mathscr{X}$}.

Theorem 16: $QH\mathscr{X} = H\mathscr{X}$.
Proof: Let $R \in QH\mathscr{X}$. Then $R = S/I$ where $S \in H\mathscr{X}$.
Every quotient of R is isomorphic to a quotient of S so $R \in H\mathscr{X}$.

Theorem 17: $I^2\mathscr{X} \subseteq I\mathscr{X}$
Proof: Let $R \in I^2\mathscr{X}$ and let A be a non-zero ideal of R.
Then there exists an ideal B of A with $A < B$ and $A/B \in I\mathscr{X}$.
Hence there exists an ideal K of A with $A/K \in \mathscr{X}$ and so $R \in I\mathscr{X}$.

Theorem 18: $N\mathscr{X} \subseteq NI\mathscr{X}$
Proof: Let $R \notin NI\mathscr{X}$. Then there exists an ideal A of R with $0 \neq R/A \in I\mathscr{X}$.
Hence there exists an ideal B of R with $0 \neq R/B \in \mathscr{X}$ and so $R \notin N\mathscr{X}$.

Theorem 19: A class of rings $\mathscr{X}$ is a radical class if and only if it is Q-closed and H-closed.
Proof: Suppose $\mathscr{X}$ is a radical class and $R \notin \mathscr{X}$.

Then $R/R_{\mathfrak{X}}$ is non-zero and has no non-zero $\mathfrak{X}$-ideal, so $R \notin H\mathfrak{X}$. Hence $\mathfrak{X}$ is H-closed.

Suppose $\mathfrak{X}$ is both Q- and H-closed.

Let $I, R/I \in \mathfrak{X}$. and suppose R/J is non-zero.

If $I \leq J$ then $R/J \cong (R/I)/(J/I) \in \mathfrak{X}$.

Otherwise $0 \neq (I + J)/J \cong I/(I \cap J) \in \mathfrak{X}$.

So $R \in H\mathfrak{X} \subseteq \mathfrak{X}$.

Suppose $R_1 \leq R_2 \leq \ldots$ is an ascending chain of $\mathfrak{X}$-ideals of a ring R and let $U = \cup R_i$.

Let $0 \neq U/J$. For each n, $(J + R_n)/J \cong R_n/(J \cap R_n) \in \mathfrak{X}$ and for some n, $(J + R_n)/J \neq 0$.

Theorem 20: If $I\mathfrak{X} = \mathfrak{X}$. then (1) $N\mathfrak{X}$ is a radical class**.**

(2) $\mathfrak{X} = (N\mathfrak{X})^{\perp}$

Proof: (1) Clearly $N\mathfrak{X}$ is Q-closed.

Suppose $R \in HN\mathfrak{X} - N\mathfrak{X}$. Since $R \notin N\mathfrak{X}$, there exists a non-zero quotient $R/K \in \mathfrak{X}$.

Since $R \in HN\mathfrak{X}$, there exists a non-zero quotient $J/K \in N\mathfrak{X}$.

But, since $R/K \in I\mathfrak{X}$, $J/K \notin N\mathfrak{X}$, a contradiction.

(2) Suppose $R \in \mathfrak{X} = I\mathfrak{X}$.

If $R_{N\mathfrak{X}} \neq 0$ then R has a non-zero homomorphic image in $\mathfrak{X}$, and so $R_{N\mathfrak{X}} \notin N\mathfrak{X}$, a contradiction. Hence $R_{N\mathfrak{X}} = 0$.

Let $R \in (N\mathfrak{X})^{\perp}$ and let K be a non-zero ideal of R.

Then $K \notin N\mathfrak{X}$. Hence there exists a non-zero quotient $K/J \in \mathfrak{X}$ and so $R \in I\mathfrak{X} = \mathfrak{X}$.

Theorem 21: If $\mathfrak{X} \subseteq \mathrm{I}\mathfrak{X}$ then $\mathrm{NI}\mathfrak{X}$ is a radical class.
Proof: $\mathfrak{X} \subseteq \mathrm{I}\mathfrak{X}$ so $\mathrm{I}^2\mathfrak{X} = \mathrm{I}\mathfrak{X}$.

Let $\mathfrak{X}$ be a class of rings. The **upper radical class** determined by $\mathfrak{X}$ is $\mathfrak{X}^{+} = \mathrm{NI}\mathfrak{X}$.
The **lower radical class** determined by $\mathfrak{X}$ is $\boldsymbol{\mathfrak{X}}^{-} = \mathrm{H}^{\infty}\mathrm{Q}\mathfrak{X}$.

Theorem 22: $\mathrm{H}^{\infty}\mathrm{Q}\mathfrak{X} = \cup \mathrm{H}^{n}\mathrm{Q}\mathfrak{X}$ is a radical class.
Proof: $\mathrm{Q}(\mathrm{H}^{\infty}\mathrm{QN}) = (\mathrm{QH})(\mathrm{H}^{\infty}\mathrm{QN}) = \mathrm{H}(\mathrm{H}^{\infty}\mathrm{QN}) = \mathrm{H}^{\infty}\mathrm{QN}$.

Theorem 23: Suppose $(\mathfrak{A}, \mathfrak{B})$ is any partition of the class $\mathbb{S}$ of simple rings.
If $\mathfrak{X}$ is any radical class with $\mathfrak{A} = \mathbb{S} \cap \mathfrak{X}$ and $\mathfrak{B} = \mathbb{S} \cap \mathfrak{X}^{\perp}$ then $\mathfrak{A}^{-} \subseteq \mathfrak{X} \subseteq \mathfrak{B}^{+}$.
Proof: $\mathrm{H}^{n}\mathrm{QN}\mathfrak{X} \subseteq \mathrm{H}^{n}\mathrm{Q}\mathfrak{X} \subseteq \mathfrak{X}$ for all n.
If $R/K \in \mathfrak{B} \subseteq \mathfrak{X}^{\perp} \cap \mathfrak{X} = \{0\}$.

Theorem 24: If $(\mathfrak{A}, \mathfrak{B})$ is any partition of $\mathbb{S}$ then $\mathfrak{B}^{-} \subset \mathfrak{A}^{+}$.
Proof: Let G be the zero ring on the Prüfer p-group (Example 5, Chapter 1).
Case I: $\mathfrak{A}$ contains a zero ring: Therefore $\mathbb{Z}_p \in \mathfrak{A}$ for some prime p.
Let n be the smallest integer such that there exists a zero ring on a non-zero p-group that lies in $\mathrm{H}^{n}\mathfrak{B}$.
Since all non-zero quotients of a non-zero p-group are non-zero p-groups, $n = 0$.

But $\mathbb{Z}_p \notin \mathcal{B}$. Hence $H^{\infty}\mathcal{B} = H^{\infty}Q\mathcal{B} = \mathcal{B}^{-}$ contains no non-zero zero ring on a p-group.

Hence $G \notin \mathcal{B}^{-}$. If there exists a non-zero quotient $G/K \cong G \in \mathcal{A}$, we get a contradiction. Hence $G \in N\mathcal{A} \subseteq NI\mathcal{A} = \mathcal{A}^{+}$ and so $G \in \mathcal{A}^{+} - \mathcal{B}^{-}$.

Case II: A contains no zero rings: Let n be the smallest integer such that $\mathbb{Z} \in H^{n}\mathcal{A}$.

Since all non-zero ideals of $\mathbb{Z}$ are isomorphic to $\mathbb{Z}$, $n = 0$.

But $\mathbb{Z} \notin \mathcal{B}$. Hence $\mathbb{Z} \notin \mathcal{B}^{-}$.

Every quotient of $\mathbb{Z}$ is a zero ring and so is not in $\mathcal{A}$.

Hence $\mathbb{Z} \in N\mathcal{A} \subseteq NI\mathcal{A} = \mathcal{A}^{+}$, so $\mathbb{Z} \in \mathcal{A}^{+} - \mathcal{B}^{-}$.

EXERCISES FOR CHAPTER 4

Exercise 1: Is the class of zero rings a radical class? Give reasons.

SOLUTIONS FOR CHAPTER 4

Exercise 1: No, it is Q closed and $\cup$-closed, but not P-closed.
Take R = $2\mathbb{Z}_{16} = \{0, 2, 4, 6, 8, 10, 12, 14\}$ mod 16.
Take I = $4\mathbb{Z} = \{0, 4, 8, 16\}$ mod 16.
Then I is a zero ring and R/I is a zero ring but R is not, since $2.4 = 8 \neq 0$ in $\mathbb{Z}_{16}$.